

# Parameter Estimation of Generalized Exponential distribution using Markov Chain Monte Carlo Method for Informative set of priors

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## ABSTRACT

*The two-parameter generalized exponential (GE) distribution was introduced and studied quite extensively by (Gupta and Kundu (1999), (2001a) and (2001b)). It is observed that GE can be used in situation where a skewed distribution for a non-negative random variable is needed. The two parameters of a GE distribution represent the scale and the shape parameters and because of the scale and shape parameters, it has quite a bit of flexibility to analyze any positive real data. Due to the simple structure of its distribution function, the GE can be used quite effectively in analyzing any lifetime data, especially in the presence of censoring or if the data is grouped. It has increasing as well as decreasing failure rate depending on the shape parameter.*

*In this paper, we analyze and estimate the parameters of the generalized exponential (GE) distribution based on complete sample. The Markov Chain Monte Carlo (MCMC) method is used to compute the Bayes estimate of the model parameters and also assumed that the parameters have informative set of priors and they are independently distributed. The maximum-likelihood estimation (MLE) is the most used method for parameter estimation; MCMC has recently emerged as a good alternative. We also compute the maximum likelihood estimate and associated confidence intervals to compare the performance of the Bayes estimators with the classical estimators. A module is incorporated in Open BUGS, a well established software for Bayesian analysis using Markov Chain Monte Carlo (MCMC) methods, for the estimation of parameters and construction of associated probability intervals. We also develop a module dgen ( $\alpha, \lambda$ ) which is written in component Pascal, enable to perform full Bayesian analysis of generalized exponential distribution. The softrelia package in the R is used for the study of the statistical properties of the model and to estimate the parameters of the model for used data set. The proposed methodology can be applied for empirical modeling, which*

*includes estimation of parameters, model validation and comparison. A real data set is considered for illustration purpose under uniform priors*

**Keywords:** Generalized Exponential (GE) Distribution, parameter estimation, informative set of priors, Maximum likelihood Estimation (MLE), Bayesian Estimation in Open Bugs, Markov Chain Monte Carlo (MCMC) Method, etc.

## INTRODUCTION

Exponential distribution play a central role in analyses of lifetime or survival data, in part because of their convenient statistical theory, their important 'lack of memory' property and their constant hazard rates. In circumstances where the one-parameter family of exponential distributions is not sufficiently broad, a number of wider families such as the gamma, Weibull and lognormal distributions are in common use. Adding parameters to a well-established family of distributions is a time honored device for obtaining more flexible new families of distributions.

The probability density function of GE is written as

$$f(x; \alpha, \lambda) = \alpha \lambda e^{-\lambda x} \left\{ 1 - e^{-\lambda x} \right\}^{\alpha-1} \quad ; (\alpha, \lambda) > 0, x > 0 \quad (1.1)$$

The typical generalized exponential density functions for different values of  $\alpha$  and for  $\lambda = 1$  shows that the density function of the generalized exponential distribution can take different shapes. It is a unimodal density function and for fixed scale parameter as the shape parameter increases it is becoming more and more symmetric. It is unimodal for  $\alpha > 1$  and it is reversed 'J' shaped for  $\alpha < 1$ . The density function of the generalized exponential distribution is log-convex if  $\alpha < 1$  and log-concave if  $\alpha > 1$ .

In this paper, the Markov chain Monte Carlo (MCMC) method is used to estimate the parameters of a generalized exponential distribution based on a complete sample. A procedure is developed to estimate the scale and shape parameters of the generalized exponential distribution using Markov Chain Monte Carlo (MCMC) simulation method in Open BUGS. A module (code) is incorporated in Open BUGS. The MCMC methods have been shown to be easy to implement computationally, the estimates always exist and are statistically consistent, and their probability intervals are convenient to construct. The R functions are developed to study the statistical properties of the distribution and the output analysis of MCMC samples generated from Open BUGS. One real data set has been analyzed for illustration.

The rest of the paper is organized as follow:

In section 2, we carry out the model analysis by discussing various function of GE distribution. In section 3; we discuss the MLE and information matrix. We give a brief description of model validation by taking a real data set in section 4. The Bayesian estimation in Open BUGS and its implementation under informative set of priors which include convergence diagnostic, visual summary and numerical summary is discussed in

section 5. In section 6, we have made comparison with MLE by way of plotting different graph. The conclusion is given in section 7.

**MODEL ANALYSIS**

The exponential distribution is generated by a method of introducing a parameter to expand the family of one-parameter exponential distribution.

Let the reliability/survival function is given by

$$R(x; \alpha, \lambda) = 1 - \left\{ 1 - e^{-\lambda x} \right\}^\alpha ; (\alpha, \lambda) > 0, x > 0 \tag{2.1}$$

The R function sgen.exp ( ) computes the reliability/ survival function.

The cumulative distribution function of generalized exponential (GE) distribution

$$F(x; \alpha, \lambda) = \left\{ 1 - e^{-\lambda x} \right\}^\alpha ; (\alpha, \lambda) > 0, x > 0 \tag{2.2}$$

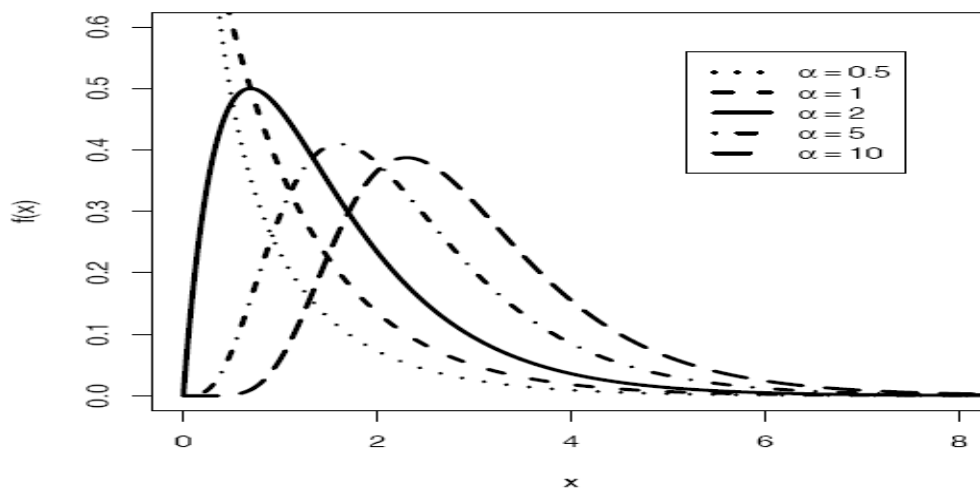
where  $\alpha > 0$  and  $\lambda > 0$  are the shape and scale parameters, respectively.

The probability density function (pdf) associated with (2.2) is given by

$$f(x; \alpha, \lambda) = \alpha \lambda e^{-\lambda x} \left\{ 1 - e^{-\lambda x} \right\}^{\alpha-1} ; (\alpha, \lambda) > 0, x > 0 \tag{2.3}$$

We denote the generalized exponential with parameters  $\alpha$  and  $\lambda$  as  $X \sim GE(\alpha, \lambda)$

The R functions dgen.exp ( ) and pgen.exp ( ) can be used for the computation of pdf and cdf, respectively. Some of the typical generalized exponential density functions for different values of  $\alpha$ , and for  $\lambda = 1$  are shown in Figure 1.



**Figure 1:** Plots of the probability density function for  $\lambda = 1$  and different values of  $\alpha$

It is clear from the Figure 1 that the density function of the generalized exponential distribution can take different shapes.

The GE  $(\alpha, \lambda)$  distribution is unimodal with mode at

$$\text{Mode} = \frac{1}{\lambda} \log \alpha \quad ; \alpha > 1 \tag{2.4}$$

And it is reverse 'J' shaped if  $\alpha, < 1$ . The mean, median, variance and mode all are non-linear functions of the shape parameter, and as the shape parameter goes to infinity, all of them tend to infinity.

$$\text{Mean} = \frac{1}{\lambda} \{ \psi(\alpha + 1) - \psi(1) \}; \tag{2.5}$$

$$\text{Variance} = \frac{1}{\lambda^2} \{ \psi'(1) - \psi'(\alpha + 1) \} \tag{2.6}$$

$$\text{Median} = -\frac{1}{\lambda} \log \left\{ 1 - (0.5)^{1/\alpha} \right\}. \tag{2.7}$$

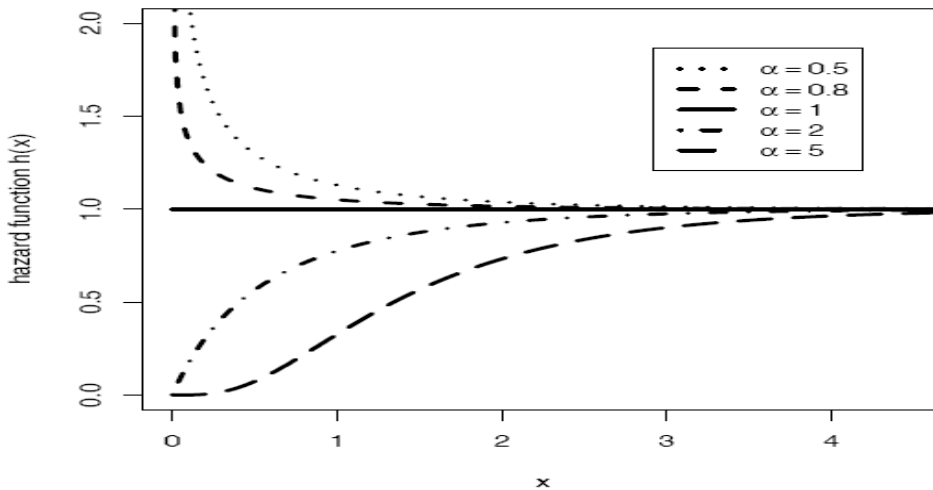
where  $\psi(\cdot)$  is the digamma function and  $\psi'(\cdot)$  is its derivative. For large values of  $\alpha$ , the mean, median and mode all are approximately equal to  $\log \alpha$ .

**The hazard function**

The hazard function is defined as

$$h(x; \alpha, \lambda) = \frac{\alpha \lambda e^{-\lambda x} \left\{ 1 - e^{-\lambda x} \right\}^{\alpha-1}}{1 - \left\{ 1 - e^{-\lambda x} \right\}^{\alpha}} \tag{2.8}$$

and the associated R function is  $hgen.exp(\cdot)$ . Some of the typical GE hazard functions for different values of  $\alpha$ .and for  $\lambda=1$  are depicted in Figure 2.



**Figure 2:** Plots of the hazard function for different values of  $\alpha$  and  $\lambda = 1$

The cumulative hazard function  $H(x)$  defined as

$$H(x) = -\{1 - \log F(x)\} \quad (2.9)$$

It can be obtained with the help of `pgen.exp ( )` function by choosing arguments `lower.tail=FALSE` and `log.p =TRUE`. i.e. `pgen.exp(x, alpha, lambda, lower.tail=FALSE, log.p=TRUE)`

The survival function (s.f.) and the conditional survival of  $X$  are defined by

$$R(x) = 1 - F(x)$$

$$\text{And } R(x | t) = \frac{R(x + t)}{R(x)}, \quad t > 0, x > 0, R(\cdot) > 0, \quad (2.10)$$

respectively, where  $F(\cdot)$  is the cdf of  $X$ . Similarly to  $h(x)$  and  $FRA(x)$

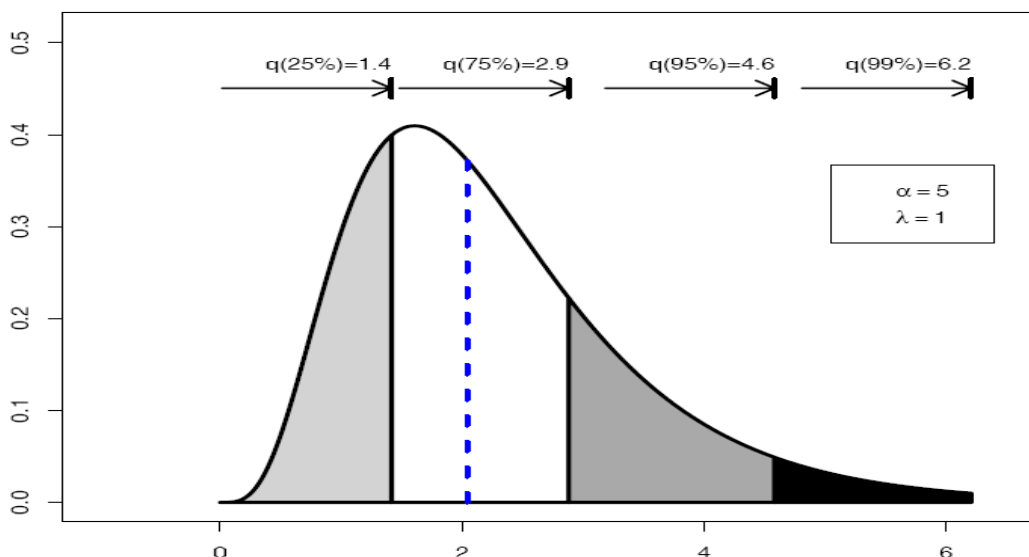
The  $R$  functions `hra.gen.exp ( )` and `crf.gen.exp ( )` are used for the computation of failure rate average (fra) and conditional survival function (crf), respectively. Graphically, the five numbers are often displayed as a **box plot**.

The quantile function is given by

$$x_q = -\frac{1}{\lambda} \log \left\{ 1 - q^{1/\alpha} \right\} ; 0 < q < 1. \quad (2.11)$$

The computation of quantiles the  $R$  function `qgen.exp ( )` can be used.

Figure 3 illustrates the use of `qgen.exp ( )` function.



**Figure 3:** An illustration of the quantiles for  $\lambda = 1$  and  $\alpha = 5$ , vertical line Indicates the median.

**Maximum Likelihood Estimation (MLE) and Information Matrix**

Let  $\underline{x}=(x_1, \dots, x_n)$  be a random sample of size  $n$  from  $GE(\alpha, \lambda)$ , then the log-likelihood function  $L(\alpha, \lambda)$  can be written as;

$$L(\alpha, \lambda) = n \log \alpha + n \log \lambda - \lambda \sum_{i=1}^n x_i + (\alpha - 1) \sum_{i=1}^n \log \{1 - e^{-\lambda x_i}\} \tag{3.1}$$

Therefore, to obtain the MLE's of  $\alpha$  and  $\lambda$  we can maximize (3.1) directly with respect to  $\alpha$  and  $\lambda$  or we can solve the following two non-linear equations using iterative procedure:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log \{1 - e^{-\lambda x_i}\} = 0 \tag{3.2}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i + (\alpha - 1) \sum_{i=1}^n \frac{x_i e^{-\lambda x_i}}{\{1 - e^{-\lambda x_i}\}} = 0 \tag{3.3}$$

Asymptotic Confidence intervals based on MLE

Since the MLEs of the unknown parameters  $\theta= (\alpha, \lambda)$  cannot be in closed forms, it is not easy to derive the exact distributions of the MLEs. We can derive the asymptotic confidence intervals of these parameters when  $\alpha > 0$ , and  $\lambda > 0$ . The large sample approach is to assume that the MLE  $(\hat{\alpha}, \hat{\lambda})$  are approximately bivariate normal with mean  $(\alpha, \lambda)$  and covariance matrix  $I_0^{-1}$ , (Lawless (2003)), where  $I_0^{-1}$  is the inverse of the observed information matrix

$$\begin{aligned}
 I_0^{-1} &= \begin{pmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} \Big|_{\hat{\alpha}, \hat{\lambda}} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \Big|_{\hat{\alpha}, \hat{\lambda}} \\ -\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} \Big|_{\hat{\alpha}, \hat{\lambda}} & -\frac{\partial^2 \ln L}{\partial \lambda^2} \Big|_{\hat{\alpha}, \hat{\lambda}} \end{pmatrix}^{-1} = \left( -H \Big|_{(\hat{\alpha}, \hat{\lambda})} \right)^{-1} \\
 &= \begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\lambda}) \\ \text{cov}(\hat{\alpha}, \hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix} \tag{3.4}
 \end{aligned}$$

The above approach is used to derive the 100(1 - γ/2) % confidence intervals of the parameters θ = (α, λ) as in the following forms

$$\hat{\alpha} \pm z_{\gamma/2} \sqrt{\text{Var}(\hat{\alpha})} \quad \text{And} \quad \hat{\lambda} \pm z_{\gamma/2} \sqrt{\text{Var}(\hat{\lambda})} \tag{3.5}$$

Here,  $z_{\gamma/2}$  is the upper (γ/2) th percentile of the standard normal distribution.

**Computation of Maximum likelihood (ML) estimation**

We consider the real data set reported in (Lawless, 2003, page 228) presented here arose in test on endurance of deep groove ball bearing. The data represented are the number of million revolutions before failure of each of the 23 ball bearing in the life test. The direct maximization of log-likelihood function given in (3.1) using Newton-Raphson method in R gives, the ML estimates and standard error. The 95% confidence interval is computed using (3.4) and (3.5).

The Table 1. Shows the ML estimates, standard error (SE) and 95 % Confidence Intervals parameters alpha and lambda

**Table 1 : ML estimate, standard error and 95% confidence interval**

Parameter	MLE	Std. Error	95% Confidence Interval
alpha	5.28321	2.04816	(1.26881, 9.29761)
lambda	0.03230	0.00642	(0.01971, 0.04488)

**Model Validation**

Most statistical methods assume an underlying distribution (model) in the derivation of their results. However, when we assume that the data follow a specific distribution, we are making an assumption. If such a model does not hold, then the conclusions from such analysis may be invalid. For example, the confidence levels of the confidence intervals or hypotheses tests implemented may be off. Although hazard plotting and the other graphical methods can guide the choice of the parametric distribution, one cannot of course be sure that the proper model has been selected. Hence model verification is still necessary to check whether we

have achieved the goal of choosing the right model. In this section we outline some of the methods used to check model appropriateness.

There are two graphical methods widely used for checking whether a fitted model is in agreement with the data. In other words, we measure the quality of the fitted model. These methods are Quantile-Quantile (Q-Q) plots and the Probability-Probability (P-P) plots in model validation.

**The Q-Q Plots**

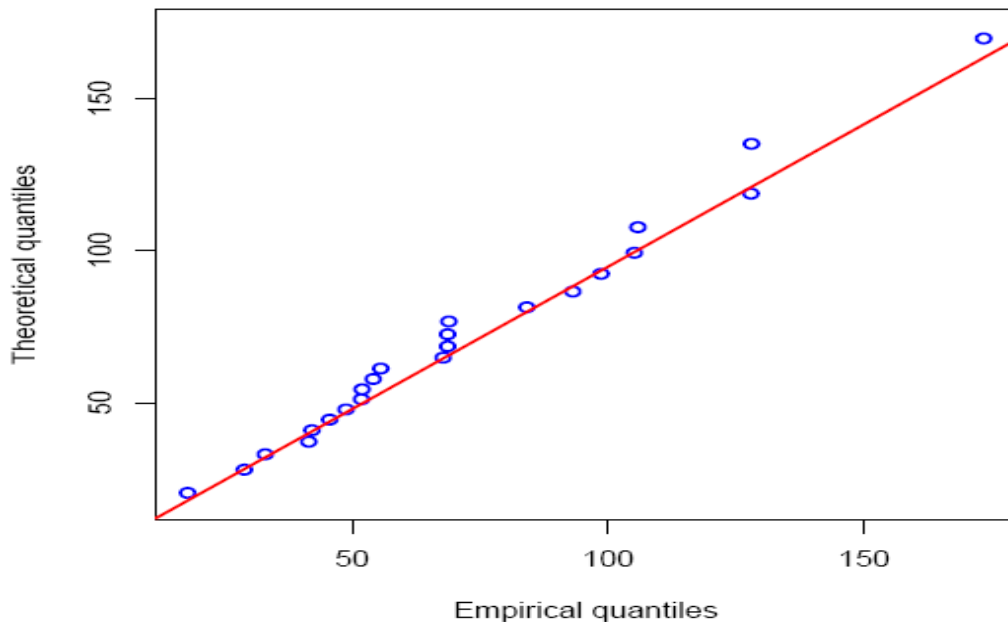
Let  $\hat{F}(x)$  be an estimate of  $F(x)$  based on  $x_1, x_2, \dots, x_n$ . The scatter plot of the points

$\hat{F}^{-1}(p_{i:n})$  versus  $x_{i:n}$ ,  $i = 1, 2, \dots, n$ , is called a Q-Q plot.

Thus, the Q-Q plots show the estimated versus the observed quantiles. If the model fits the data well, the pattern of points on the Q-Q plot will exhibit a 45-degree straight line. Note that all the points of a Q-Q plot are inside the square

$$\left[ \hat{F}^{-1}(p_{1:n}), \hat{F}^{-1}(p_{n:n}) \right] \times \left[ x_{1:n}, x_{n:n} \right].$$

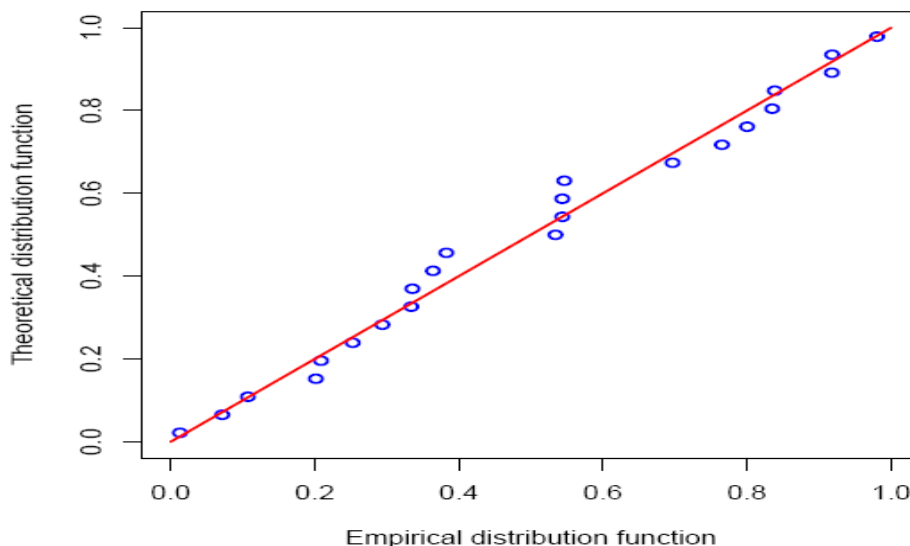
The corresponding R function `qq.gen.exp()`. It can be seen from the straight line pattern in Figure 4; the GE fits the data very well.



**Figure 4 :** Quantile-Quantile (Q-Q) plot using MLEs as estimate.



**The P-P Plots**



**Figure 5:** Probability-Probability (P-P) plot using MLEs as estimate

Let  $x_1, x_2, \dots, x_n$  be a sample from a given population with estimated cdf  $\hat{F}(x)$ . The scatter plot of the points

$\hat{F}(x_{i:n})$  versus  $p_{i:n}$ ,  $i = 1, 2, \dots, n$ , is called a P-P plot. If the model fits the data well, the graph will be close to the 45-degree line. We see that all the points in the P-P plot are inside the unit square  $[0, 1] \times [0, 1]$ . The corresponding R function is `pp.gen.exp()`. It can be seen from the straight line pattern in Figure 5 the GE distribution fits the data very well.

**Bayesian Estimation in Open BUGS**

Statistical models must be described before they can be used. A language to describe statistical models is needed by both the users of the model and the software that makes inference about the model. The language should be a formal language with well defined rules which can be processed automatically as far as possible to make inferences about the model. A small change to a model should not lead to a large change in the way that model is described

The most widely used piece of software for applied Bayesian inference is the WinBUGS package, distributed by the MRC Biostatistics Unit at Cambridge (Spiegelhalter, Thomas, Best, and Lunn 2004).

A module **dgen.exp (alpha, lambda)** is written in component Pascal to perform full Bayesian analysis of generalized exponential (GE) model into Open BUGS using the method described in Thomas (2007).

**Implementation of Module - dgen.exp (alpha, lambda) :**

The developed module is implemented to obtain the Bayes estimates of the GE distribution using MCMC method. The main function of the module is to generate MCMC sample from posterior distribution for given set of priors.

**Uniform prior**

It frequently happens that the researcher knows in advance that the probable values of  $\theta$  lie over a finite range  $[a, b]$  but has no strong opinion about any subset of values over this range. In such a case a uniform distribution over  $[a, b]$  may be a good approximation of the prior distribution, its p.d.f. is given by

$$\pi(\Theta) = \begin{cases} \frac{1}{b - a} & ; 0 < a < \theta < b \\ 0 & ; \text{otherwise} \end{cases}$$

Where a and b are the parameters.

**Bayesian Analysis under Informative set of Priors**

**Model**

```
{
For (i in 1: N)
{
x[i] ~ dgen.exp(alpha, lambda)
}
# Prior distributions of the Model parameters
# Uniform prior for alpha
alpha ~ dunif(0.0, 20.0)
# Uniform prior for lambda
lambda~ dunif(0.0, 2.0)
}
```

**Data**

```
list( N=23, x = c(17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56,
67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40))
```

**Initial values :**

```
# Chain 1
list(alpha=4.0, lambda=0.01)
```

# Chain 2

list(alpha=8.0, lambda=0.1)

We run the model to generate two Markov Chains at the length of 40,000 with different starting points of the parameters. The convergence is monitored using trace and ergodic mean plots, we find that the Markov Chain converge together after approximately 2000 observations. Therefore, burning of 5000 samples is more than enough to erase the effect of starting point (initial values). Finally, samples of size 7000 are formed from the posterior by picking up equally spaced every fifth outcome, i.e. thin=5, starting from 5001. This is done to minimize the auto correlation among the generated deviates.

Therefore, we have the posterior sample  $\{\alpha_{1i}, \lambda_{1i}\}$ ,  $i = 1, \dots, 7000$  from chain 1 and  $\{\alpha_{2i}, \lambda_{2i}\}$ ,  $i = 1, \dots, 7000$  from chain 2.

The chain 1 is considered for convergence diagnostics plots. The visual summary is based on posterior sample obtained from chain 2 whereas the numerical summary is presented for both the chains.

Convergence Diagnostics:

History (Trace) plot

Before examining the parameter estimates or performing other inference, it is a good idea to look at plots of the sequential (dependent) realizations of the parameter estimates and plots thereof. We have found that if the Markov chain is not mixing well or is not sampling from the stationary distribution, this is usually apparent in sequential plots of one or more realizations. The sequential plot of parameters is the plot that most often exhibits difficulties in the Markov chain. Figure 6 shows the sequential realizations of the parameters of the model. In this case Markov chain seems to be mixing well enough and is likely to be sampling from the stationary distribution.

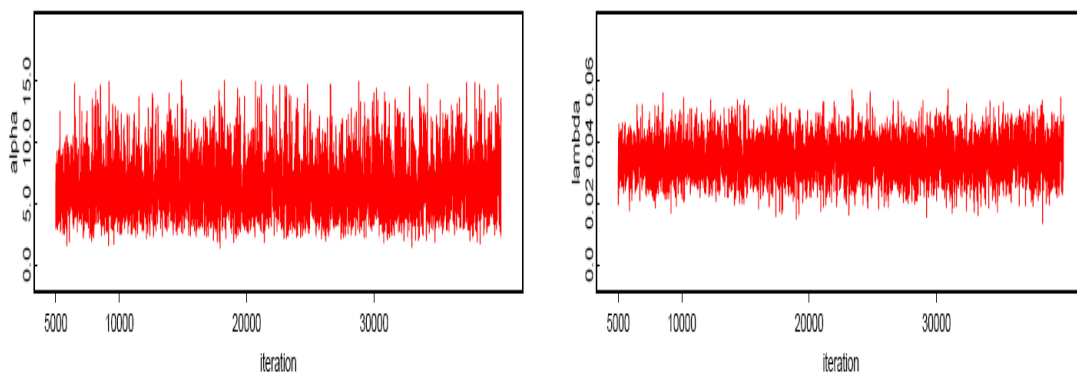
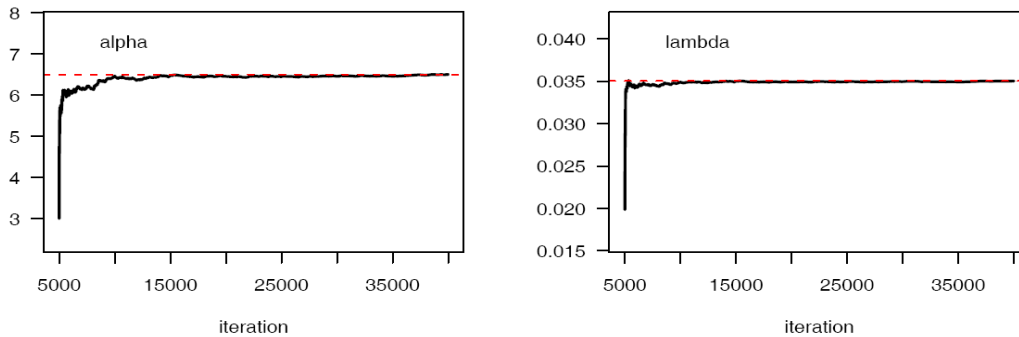


Figure 6 : Sequential realization of the parameters  $\alpha$  and  $\lambda$ .

The plot looks like a horizontal band, with no long upward or downward trends, so we have evidence that the chain has converged.

**Running Mean (Ergodic mean) Plot**

Generate a time series (iteration number) plot of the running mean for each parameter in the chain. The running mean is computed as the mean of all sampled values up to and including that at a given iteration.

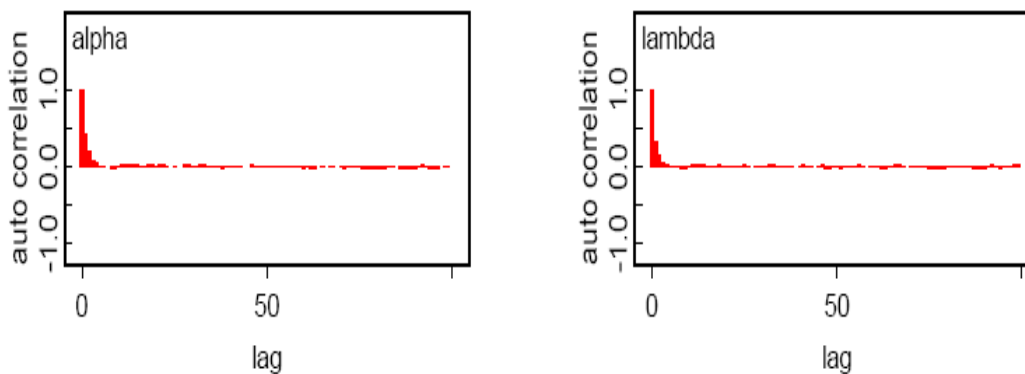


**Figure 7:** The Ergodic mean plots for alpha and lambda.

The convergence pattern based on ergodic averages is shown in Figure 7 which indicates the convergence of the chain

**Autocorrelation**

The graph shows that the correlation is almost negligible. We may conclude that the samples are independent.



**Figure 8:** The autocorrelation plots for alpha and lambda.

**Numerical Summary**

In Table 2. We have considered various quantities of interest and their numerical values based on MCMC sample of posterior characteristics for generalized exponential(GE) distribution under uniform priors. The Highest probability density (HPD) intervals are computed the algorithm described by Chen and Shao (1999) under the assumption of unimodal marginal posterior distribution.

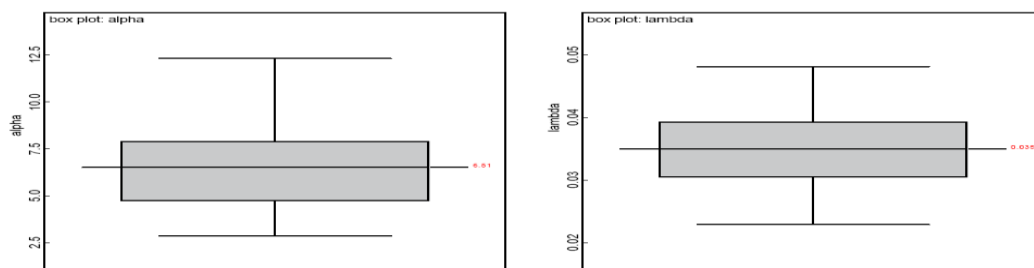
**Table 2:** MCMC sample based posterior characteristics for Generalized Exponential distribution

Characteristics	Chain 1		Chain 2	
	alpha	lambda	alpha	lambda
Mean	6.503	0.03503	6.541	0.03502
Standard Deviation(SD)	2.393	0.006415	2.457	0.006511
Monte Carlo(MC) error	0.04831	0.000111	0.04731	0.000110
Minimum	1.441	0.01375	1.372	0.0123
First Quartile (Q <sub>1</sub> )	4.706	0.03052	4.724	0.03046
Median	6.162	0.03487	6.144	0.03485
Third Quartile (Q <sub>3</sub> )	7.895	0.03946	7.939	0.03938
Maximum	14.99	0.05715	14.98	0.05788
Mode	5.02823	0.03444	5.25164	0.03464
2.5th Percentile(P <sub>2.5</sub> )	2.852	0.02304	2.866	0.02292
97.5th Percentile(P <sub>97.5</sub> )	12.26	0.04794	12.68	0.04833
95% Credible Interval	(2.852, 12.26)	(0.02304, 0.04794)	(2.866, 12.68)	(0.02292, 0.04833)
95% HPD Credible Interval	(2.322, 11.260)	(0.02244, 0.04704)	(2.411, 11.550)	(0.02247, 0.04769)

**Visual summary**

**Box plots:**

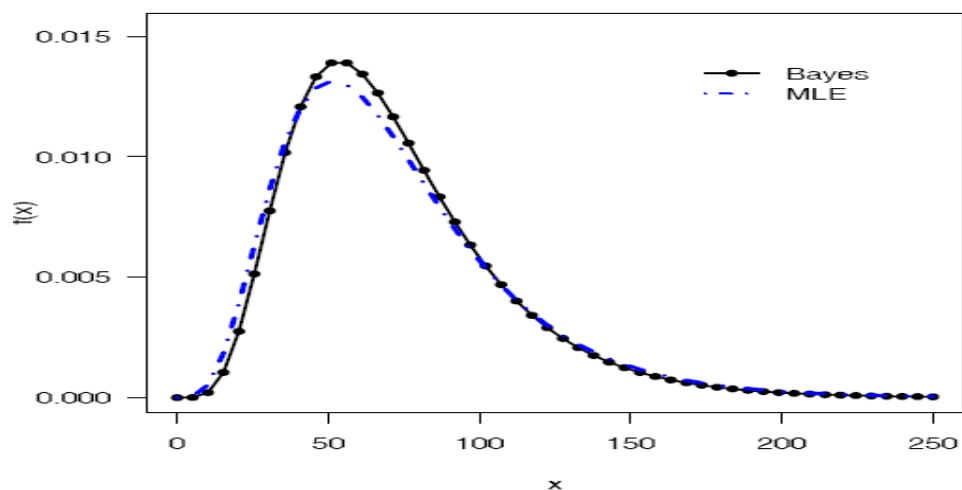
The boxes represent inter-quartile ranges and the solid black line at the (approximate) centre of each box is the mean; the arms of each box extend to cover the central 95 percent of the distribution - their ends correspond, therefore, to the 2.5% and 97.5% quantiles.



**Figure 9:** The box plots for alpha and lambda.

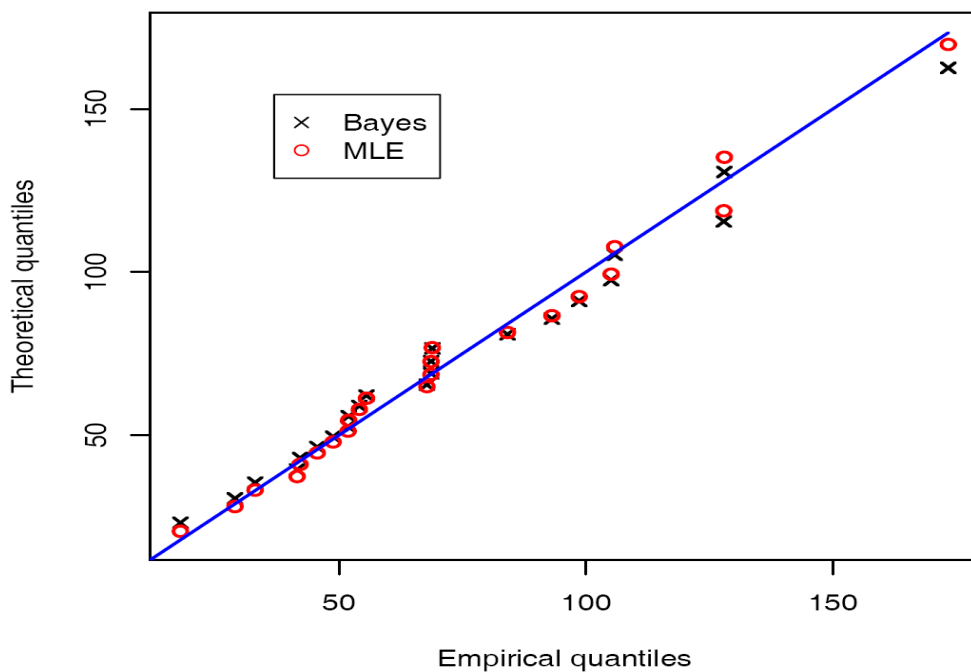
**Comparison with MLE**

For the comparison with MLE we have plotted three graphs. In Figure 10 the density functions  $f(x; \hat{\alpha}, \hat{\lambda})$  using MLEs and Bayesian estimates, computed via MCMC samples under uniform priors, are plotted.



**Figure 10:** The density functions  $f(x; \hat{\alpha}, \hat{\lambda})$  using MLEs and Bayesian estimates, Computed via MCMC samples under uniform priors.

The Figure 11 represent the Quantile-Quantile (QQ) plot of empirical quantiles and theoretical quantiles computed from MLE and Bayes estimates



**Figure 11:** Quantile-Quantile (QQ) plot of empirical quantiles and theoretical quantiles computed from MLE and Bayes estimates.

It is clear from the Figures 11; the MLEs and the Bayes estimates with respect to the uniform priors are quite close and fit the data very well.

## CONCLUSION

The developed methodology for MLE and Bayesian estimation has been demonstrated on a real data set when both the parameters alpha (shape) and lambda (scale) of the GE distribution are unknown under informative set of independent priors. The bayes estimates of the said priors have been obtained under squared error, absolute error and zero-one loss functions. A five point summary Minimum (x),  $Q_1$ ,  $Q_2$ ,  $Q_3$ , Maximum (x) has been computed. The symmetric Bayesian credible intervals and Highest Probability Density (HPD) intervals have been constructed. With the use of graphical representations the intent is that one can gain a perspective of various meanings and associated interpretations.

The MCMC method provides an alternative method for parameter estimation of the GE distributions. It is more flexible when compared with the traditional methods such as MLE method. Moreover, 'exact' probability intervals are available rather than relying on estimates of the asymptotic variances. Indeed, the MCMC sample may be used to completely summarize posterior distribution about the parameters, through a kernel estimate. This is also true for any function of the parameters such as hazard function, mean time to failure etc. The MCMC procedure can easily be applied to complex Bayesian modeling relating to GE model.

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